

A NEW CLASS OF NON-IDENTIFIABLE SKEW SYMMETRIC TENSORS

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ABSTRACT. We prove that the generic element of the fifth secant variety $\sigma_5(Gr(\mathbb{P}^2, \mathbb{P}^9)) \subset \mathbb{P}(\wedge^3 \mathbb{C}^{10})$ of the Grassmannian of planes of \mathbb{P}^9 has exactly two decompositions as sum of five projective classes of decomposable skew-symmetric tensors. We show that this is the only non-identifiable case among the non-defective secant varieties $\sigma_s(Gr(\mathbb{P}^k, \mathbb{P}^n))$ for any $n < 14$. In the same range for n we classify all the weakly defective and all tangentially weakly defective secants varieties of any Grassmannians. We also show that the dual variety $(\sigma_3(Gr(\mathbb{P}^2, \mathbb{P}^7)))^\vee$ of the variety of 3-secant planes of the Grassmannian of $\mathbb{P}^2 \subset \mathbb{P}^7$ is $\sigma_2(Gr(\mathbb{P}^2, \mathbb{P}^7))$ the variety of bi-secant lines of the same Grassmannian.

INTRODUCTION

Let $X \subset \mathbb{P}^n$ be any reduced, irreducible projective variety defined over \mathbb{C} . A point $t \in \mathbb{P}^n$ has X -rank equal to r if r is the minimum integer for which there exist r points $x_1, \dots, x_r \in X$ such that

$$(1) \quad t \in \langle x_1, \dots, x_r \rangle$$

$\langle x_1, \dots, x_r \rangle \simeq \mathbb{P}^{r-1}$ denotes the projective linear span of the x_i 's. We will also say that in this case $\{x_1, \dots, x_s\}$ is a *decomposition* of t . The Zariski closure of the set $\{t \in \mathbb{P}^n \mid X\text{-rank}(t) = r\}$ is the so called *r -secant variety* $\sigma_r(X)$ of X . There is an expected dimension for $\sigma_r(X)$ that is $\text{expdim } \sigma_r(X) = \min\{r(\dim X + 1) - 1, \dim \langle X \rangle\}$. The actual dimension of $\sigma_r(X)$ can be smaller than the expected as it can be computed by Terracini's Lemma (see e.g. [20, 2]). When this happens we say that X is *r -defective* with *r -defect* $\delta = \text{expdim } \sigma_r(X) - \dim \sigma_r(X)$.

The *r -th secant degree* of X is the number of \mathbb{P}^{r-1} 's containing the generic element $t \in \sigma_r(X)$ and that are r -secant to X as in (1). Regarding the r -th secant degree, when the dimension of $\sigma_r(X)$ is not the expected one, it is infinity.

The variety X is said to be *r -identifiable* if the r -th secant degree of X is equal to 1.

Moreover, X is said to be *perfect* if $(\dim X + 1)$ divides $n + 1$. In this case we expect a finite number of decompositions also for a generic $t \in \mathbb{P}^n$. Note that the r -th secant degree is well defined even for the generic value, in the perfect case. The generic identifiability in a perfect case is rare, but when it happens it implies that we have a *canonical form* (see e.g. [21, 18]). Having a canonical form means that the generic element $t \in \mathbb{P}^n = \langle X \rangle$ can be written in a unique way as sum of r elements on X if $\sigma_r(X)$ is the first secant variety filling \mathbb{P}^n . The most celebrated case when this situation appears is the famous Pentahedral Theorem of Sylvester: the generic quaternary cubic can be written in a unique way as sum of 5 cubic forms.

Let H be a general hyperplane section of X tangent at r general points $t_1, \dots, t_r \in X$ with r sub-generic, the *contact locus* of H , is the union of the irreducible components of $\text{Sing}(H)$ containing t_1, \dots, t_r . Remark that since t_1, \dots, t_r are general points, then the contact locus is equidimensional. Now X is *r -weakly defective* if the general $(r+1)$ -tangent hyperplane to X has a contact locus of positive dimension (these concepts were introduced in [8]).

It is worth to remark that finding a contact locus of positive dimension is not enough for claiming the non identifiability of the generic element (while the viceversa is true: if the contact locus is zero-dimensional then we have the uniqueness of the decomposition). Nevertheless there is a more refined notion that is closer related to identifiability, namely the *tangentially weakly defectiveness*. Let $p_1, \dots, p_r \in X$ be r general points of a variety X ; the r -tangentially contact locus of X is the set of points $\{p \in X \mid T_p X \subset \langle T_{p_1} X, \dots, T_{p_r} X \rangle\}$. A variety X is said to be *r -tangentially weakly defective* if the r -tangentially contact locus has positive dimension. If X is not r -tangentially weakly defective then we have the identifiability of the generic element of $\sigma_r(X)$ ([9, Proposition 2.4]). This is not an “if and only if” criterion, but still the r -tangentially contact locus of X gives the right information on the number of decompositions of the generic element of $\sigma_r(X)$: in fact the r -secant degree is equal to the degree of the r -tangentially contact locus of X (cfr [8]).

In this paper we focus on the case of X being a Grassmann variety in its Plücker embedding $Gr(\mathbb{P}^k, \mathbb{P}^n) \subset \mathbb{P}(\bigwedge^{k+1} \mathbb{C}^{n+1})$. It parameterizes projective classes of skew-symmetric tensors that can be written as $v_1 \wedge \dots \wedge v_{k+1}$ with $v_i \in \mathbb{C}^n$ for $i = 1, \dots, k+1$. Therefore we will say that $t \in \mathbb{P}(\bigwedge^{k+1} \mathbb{C}^{n+1})$ has skew-symmetric rank r if it belongs to a \mathbb{P}^{r-1} which is r -secant to $Gr(\mathbb{P}^k, \mathbb{P}^n)$, with minimal r . Since we will always deal with skew-symmetric tensors, there won't be any risk of confusion if we will simply say that such a t has *rank* r .

On defective secant varieties to Grassmann varieties there is an open conjecture (stated independently in [1, 4, 7]) that says that defective Grassmannians occur only for $Gr(\mathbb{P}^1, \mathbb{P}^n)$ for any n , $Gr(\mathbb{P}^2, \mathbb{P}^6)$, $Gr(\mathbb{P}^3, \mathbb{P}^7)$, $Gr(\mathbb{P}^2, \mathbb{P}^8)$ (see also [6] for a recent proof for $\sigma_s(Gr(\mathbb{P}^k, \mathbb{P}^n))$ with $s \leq 12$).

A classical result due to C. Segre (see [19]) shows that $Gr(\mathbb{P}^2, \mathbb{P}^5)$ has the 2-nd secant degree equal to 1, i.e. there is a canonical form for the generic element in $\mathbb{P}(\bigwedge^3 \mathbb{C}^6)$ that is therefore of type $[v_1 \wedge v_2 + w_1 \wedge w_2]$ with $v_i, w_i \in \mathbb{C}^6$, $i = 1, 2$. After the example of C. Segre, the next interesting perfect cases are $Gr(\mathbb{P}^3, \mathbb{P}^8)$ and $Gr(\mathbb{P}^4, \mathbb{P}^8)$ (dual to each other) for which the secant degree is unknown. In order to have a numerical evidence on the behavior of these two cases we firstly made use of Bertini ([3]): it is possible to show that the decompositions of the generic element in $\mathbb{P}(\bigwedge^4 \mathbb{C}^9)$ as a sum of 6 elements in $Gr(\mathbb{P}^3, \mathbb{P}^8)$ are a finite number. The number of decompositions that we found with Bertini is high (more than 7000). The software Bertini is a good tool to have a numerical evidence on the order of magnitude of the number of the decompositions, but we did not pursue this path since, having found such a big number of decompositions, we won't ever discover the precise amount of them only using Bertini (see [16] for a first application of homotopy continuation method with Bertini to the study of tensors identifiability, and [5] for its application to a new numerical algorithm for tensor decomposition). What we can claim is the following: since we are in a perfect case, the fact that we found at least two numerically different decompositions, implies that the generic

element of $\bigwedge^4 \mathbb{C}^9$ is not 6-identifiable. In fact in any perfect case the map from the abstract r -secant variety $S_r = \{(x_1, \dots, x_r; t) \in X^r \times \mathbb{P}(\langle X \rangle) \mid t \in \langle x_1, \dots, x_r \rangle\}$ to the r -secant variety itself is generically finite, therefore, by Zariski's main theorem (see [22]), if the map was birational it would have connected fibers, but one could check, by computing the dimension of the tangent space, that at least one of the two different decompositions is an isolated point of the fiber. As already anticipated, we included here these considerations for sake of completeness but won't work out this argument within the manuscript.

In this paper we firstly compute the contact locus of all the highest secant varieties of the Grassmannians $Gr(\mathbb{P}^k, \mathbb{P}^n)$ that does not fill the ambient space for $n+1 \leq 14$. Secondly we find that, among the non-defective ones, the only ones having positive dimensional contact locus are $\sigma_3(Gr(\mathbb{P}^2, \mathbb{P}^7))$ and $\sigma_5(Gr(\mathbb{P}^2, \mathbb{P}^9))$. In the first case we find that the generic element of $\sigma_3(Gr(\mathbb{P}^2, \mathbb{P}^7))$ is actually identifiable, therefore this is an example of a 3-weakly-defective Grassmannian having identifiable generic elements. An important remark in this respect will be Proposition 1.2 where we show that the dual variety of $\sigma_3(Gr(\mathbb{P}^2, \mathbb{P}^7))$ is $\sigma_2(Gr(\mathbb{P}^2, \mathbb{P}^7))$. It will turn out that $\sigma_2(Gr(\mathbb{P}^2, \mathbb{P}^7))$, $\sigma_3(Gr(\mathbb{P}^2, \mathbb{P}^7))$ and $\sigma_5(Gr(\mathbb{P}^2, \mathbb{P}^9))$ are the only weakly-defective secant varieties being not defective for $n < 14$. The second case of $\sigma_5(Gr(\mathbb{P}^2, \mathbb{P}^9))$ is a new example for non-identifiability and it is the unique one among the non-defective cases for $n < 14$. In Proposition 1.11 we show that the generic order 3 skew-symmetric tensor of \mathbb{C}^{10} of rank 5 belongs to exactly two \mathbb{P}^4 's 5-secant to $Gr(\mathbb{P}^2, \mathbb{P}^9)$.

Our main result is Theorem 1.1 where we compute all secants degree for any Grassmannian if $n < 14$. Finally we conclude the paper with two Corollaries 1.12 and 1.13 where we classify all the weakly defective cases and all the tangentially weakly defective cases for the same range $n < 14$.

1. NEW NON-IDENTIFIABLE GRASSMANNIAN

In order to compute the contact locus for all the secant varieties of the Grassmannians $Gr(\mathbb{P}^k, \mathbb{P}^n)$ that does not fill the ambient space for $n+1 \leq 14$ we make use of Macaulay2 [14] (see the file `grascontactlocus.m2` in the ancillary material). For those computations we have used the Hessian criterion introduced in [10] (see [10, Lemma 4.3, Lemma 4.4, and Theorem 4.5]) suitably adapted to skew-symmetric tensors. We stopped to $n+1 = 14$ because, after such a value of n , the computational cost of running the program becomes too high. The main theorem of this paper is the following:

Theorem 1.1.

- (1) (a) *The Grassmannian $Gr(\mathbb{P}^2, \mathbb{P}^7)$ is 2 and 3-weakly defective and the generic elements of $\sigma_2(Gr(\mathbb{P}^2, \mathbb{P}^7))$ and of $\sigma_3(Gr(\mathbb{P}^2, \mathbb{P}^7))$ are identifiable.*
 (b) *The dual variety $\sigma_3(Gr(\mathbb{P}^2, \mathbb{P}^7))^\vee$ is $\sigma_2(Gr(\mathbb{P}^2, \mathbb{P}^7))$.*
- (2) *The Grassmannian $Gr(\mathbb{P}^k, \mathbb{P}^n)$ is r -identifiable for $n < 14$ and r sub-generic except for:*
 - (a) $\sigma_r(Gr(\mathbb{P}^1, \mathbb{P}^n))$, $2r \leq n+1$;
 - (b) $\sigma_3(Gr(\mathbb{P}^2, \mathbb{P}^6)) \simeq \sigma_3(Gr(\mathbb{P}^3, \mathbb{P}^6))$;
 - (c) $\sigma_5(Gr(\mathbb{P}^2, \mathbb{P}^9)) \simeq \sigma_5(Gr(\mathbb{P}^6, \mathbb{P}^9))$;
 - (d) $\sigma_3(Gr(\mathbb{P}^3, \mathbb{P}^7))$;

- (e) $\sigma_4(Gr(\mathbb{P}^3, \mathbb{P}^7))$;
- (f) $\sigma_4(Gr(\mathbb{P}^2, \mathbb{P}^8)) \simeq \sigma_4(Gr(\mathbb{P}^5, \mathbb{P}^8))$.

Moreover the 5th-secant degree of $Gr(\mathbb{P}^2, \mathbb{P}^9)$ is 2 (case (2c)), in all the other exceptional cases the corresponding r^{th} -secant degree of $Gr(\mathbb{P}^k, \mathbb{P}^n)$ is infinity.

Proof. Item (1) is proved in Section 1.1. Item (2c) is proved in Section 1.2. All the other cases listed above correspond to defective secant varieties (cfr. [6, 1, 4, 7]).

The fact that there are no other exception is a consequence of the fact that there are no other positive dimensional contact locus except for $\sigma_3(Gr(\mathbb{P}^2, \mathbb{P}^7))$ and $\sigma_5(Gr(\mathbb{P}^2, \mathbb{P}^9))$ among the non defective cases: clearly if X is an r -weakly defective variety then it is also $(r+k)$ -weakly defective for any $1 \leq k < \min\{s \in \mathbb{N} \mid \sigma_s(X) = \langle X \rangle\} - r$; and if X is r -identifiable then it is also $(r-k)$ -identifiable for any $0 \leq k \leq r-1$. While for $Gr(\mathbb{P}^2, \mathbb{P}^9)$ we have proved by direct computation that it is not 4-weakly defective, hence its generic element is 4-identifiable.

Finally the 2-identifiability of $Gr(\mathbb{P}^2, \mathbb{P}^6)$ and of $Gr(\mathbb{P}^3, \mathbb{P}^7)$ and the 3-identifiability of $Gr(\mathbb{P}^2, \mathbb{P}^8)$ were directly computed with Macaulay2. More precisely we found a 6 dimensional contact locus for $\sigma_2(Gr(\mathbb{P}^2, \mathbb{P}^6))$, so it is potentially weakly defective, but we computed that $Gr(\mathbb{P}^2, \mathbb{P}^6)$ is not 2-tangentially weakly defective, therefore we have the 2-identifiability for its generic element. While $Gr(\mathbb{P}^3, \mathbb{P}^7)$ is not 2-weakly defective and $Gr(\mathbb{P}^2, \mathbb{P}^8)$ is not 3-weakly defective. \square

1.1. Identifiability for the generic element of $\sigma_3(Gr(\mathbb{P}^2, \mathbb{P}^7))$. The computation that we have done with Macaulay2 [14] (see `grascontactlocus.m2` in the ancillary material) shows that $\sigma_3(Gr(\mathbb{P}^2, \mathbb{P}^7))$ has a positive dimensional contact locus, i.e. that it is weakly-defective, with “high probability”. Before investigating on the identifiability of the generic element we like to show that $Gr(\mathbb{P}^2, \mathbb{P}^7)$ is indeed 3-weakly defective. We will make use of the fact that a variety X is r -weakly defective if and only if the dimension of the dual variety to $\sigma_r(X)$ is smaller than $\dim(\mathbb{P}(X)) - r$ (see [8]). We will also say that a variety X is *dual defective* if its dual variety X^\vee is not a hypersurface.

Proposition 1.2. *The dual variety $(\sigma_3(Gr(\mathbb{P}^2, \mathbb{P}^7)))^\vee$ is $\sigma_2(Gr(\mathbb{P}^2, \mathbb{P}^7))$ and the Grassmannian $Gr(\mathbb{P}^2, \mathbb{P}^7)$ is 3-weakly defective with a 7 dimensional contact locus.*

Proof. Remark that $SL(8)$ has only a finite number of orbits on $\mathbb{P}(\bigwedge^3 \mathbb{C}^8)$. G.B. Gurevich in [15, VII, §35.4] gave the complete classification of those orbits; their dimensions are computed by D. Ž. Djoković in [11, Table I]. We retrieve this classification in our Table 1.

Notation 1.3 (for Table 1). The table is splitted vertically in two parts: on the same row we write the orbits that are dual to each other. We have checked them via a dimension count, in fact, since $SL(8)$ has only a finite number of orbits on $\mathbb{P}(\bigwedge^3 \mathbb{C}^8)$, then the dual variety of an orbit closure remains a homogeneous variety, therefore it has to be one of those classified by Gurevich in [15, VII, §35.4]. The only ambiguity exists for XV and XIX; we prove this case along the present proof. We follow the notation of [15]: in the first (5th resp.) column the numbers of the orbits are the same used by Gurevich in [15, VII, §35.4]; in the second (7th resp.) column we write the canonical form (C.F.) of an element in each orbit; in the third (8^{ve} resp.) column we write the affine dimension (D.) of the corresponding orbit and in the 4th (last resp.) column we write the variety of the orbit closure.

The notation for the canonical form used in Table 1 is the following $[abc][qrs] := a \wedge b \wedge c + q \wedge r \wedge s$ where $a, b, c, q, r, s \in \mathbb{C}^8$. Moreover, in that table “ G ” stays for $Gr(\mathbb{P}^2, \mathbb{P}^7)$; “ C ” for restricted chordal variety; “ τ ” for the tangential variety to $Gr(\mathbb{P}^2, \mathbb{P}^7)$; “ σ_i ” for $\sigma_i(Gr(\mathbb{P}^2, \mathbb{P}^7))$, $i = 2, 3$; “ $J(G, X)$ ” for the join variety among $Gr(\mathbb{P}^2, \mathbb{P}^7)$ and the variety X ; and “ S_i ” for the subspace variety $Sub_i(\wedge^3 \mathbb{C}^8) := \{t \in \mathbb{P}(\wedge^3 \mathbb{C}^8) \mid \exists \mathbb{C}^i \subset \mathbb{C}^8 \text{ s.t. } t \in \mathbb{P}(\wedge^3 \mathbb{C}^i)\}$, $i = 6, 7$.

TABLE 1. Classification of the orbits of $SL(8)$ on $\mathbb{P}(\wedge^3 \mathbb{C}^8)$. Notation is settled in Notation 1.3.

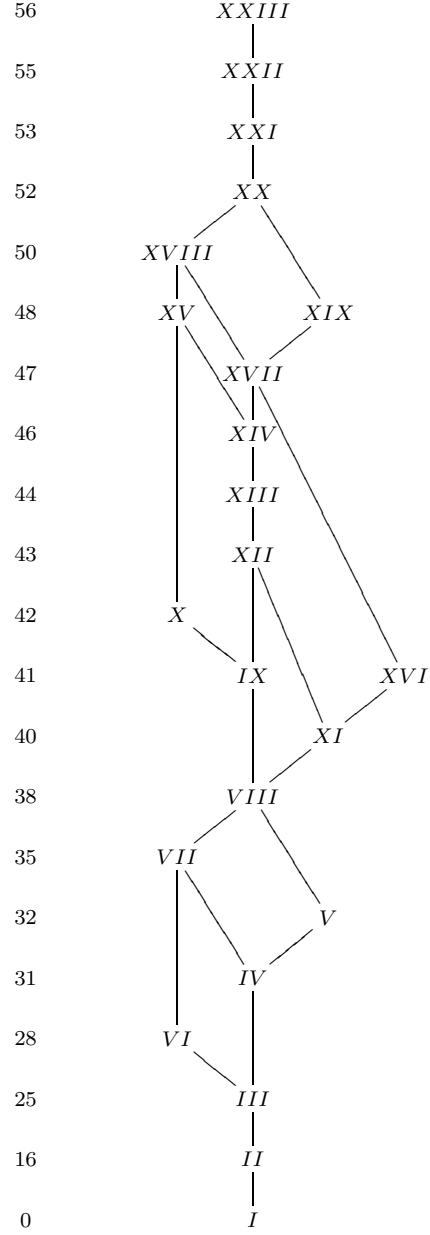
	C.F.	D.	Var.		C.F.	D.	Var.
I	$w = 0$	0		XXIII	$[abc][qrs][aqp][brp]$ $[csp][bst][crt]$	56	\mathbb{P}^{55}
II	$[qrs]$	16	G	XXII	$[abc][qrs][aqp][brp]$ $[bst][crt]$	55	G^\vee
III	$[aqp][brp]$	25	C	XXI	$[abc][qrs][aqp][bst]$	53	C^\vee
IV	$[aqr][brp][cpq]$	31	τ	XX	$[qrs][aqp][brp][csp]$ $[bst][crt]$	52	τ^\vee
V	$[abc][pqr]$	32	$\sigma_2 = S_6$	XIX	$[aqp][brp][csp][bst]$ $[crt]$	48	σ_3
VI	$[aqp][brp][csp]$	28	S_7^\vee	X	$[abc][qrs][aqp][brp]$ $[csp]$	42	S_7
VII	$[abc][prq][aps]$	35		XVIII	$[qrs][aqp][brp][bst]$ $[crt]$	50	
VIII	$[abc][qrs][aqp]$	38		XVII	$[aqp][brp][bst][crt]$	47	
IX	$[abc][qrs][aqp][brp]$	41	$J(G, C)^\vee$	XVI	$[aqp][bst][crt]$	41	$J(G, C)$
XI	$[aqp][brp][csp][crt]$	40		XV	$[abc][qrs][aqp][brp]$ $[csp][crt]$	48	
XII	$[qrs][aqp][brp][csp]$ $[crt]$	43		XIV	$[abc][qrs][aqp][brp]$ $[crt]$	46	
XIII	$[abc][qrs][aqp][crt]$	44			SELF DUAL		

For sake of completeness we include in Table 2 the containment diagram of the orbit closures of $SL(8)$ on $\mathbb{P}(\wedge^3 \mathbb{C}^8)$ (we want to thank W.A. de Graaf for his help with SLA [13] GAP4 [12] package, in drawing this diagram; anyway the same diagram is also detailed described in [11, Figure 1]).

The variety $\sigma_3(Gr(\mathbb{P}^2, \mathbb{P}^7))$ is not defective (from the dimension of the secant variety point of view), therefore its affine cone has dimension 48. Gurevich in [15] shows that $SL(8)$ generates two orbits of affine dimension 48 in $\mathbb{P}(\wedge^3 \mathbb{C}^8)$: XV and XIX (as illustrated in Table 2). One of them must be the open part of $\sigma_3(Gr(\mathbb{P}^2, \mathbb{P}^7))$. Gurevich also shows that the dual variety of the closure of XIX has affine dimension 32 and its open part is the orbit of $a \wedge b \wedge c + p \wedge q \wedge r$ (it is represented by V in Table 1), i.e. the closure of V is obviously $\sigma_2(Gr(\mathbb{P}^2, \mathbb{P}^7))$. Therefore if we prove that $\sigma_3(Gr(\mathbb{P}^2, \mathbb{P}^7))$ is the closure of XIX we are done.

Either if $\sigma_3(Gr(\mathbb{P}^2, \mathbb{P}^7))$ is the closure of XIX or of XV, we have that it is dual defective: in one case its dual variety would have affine dimension 32 and in the other 40 (in both cases the dual variety of $\sigma_3(Gr(\mathbb{P}^2, \mathbb{P}^7))$ won't be of dimension $55-3=52$). Now the point is that there is a link between the contact locus of a secant variety and its dual variety (as it is shown in [8]). More precisely: the codimension of the dual variety of a secant variety $\sigma_k(X)$ which is not defective but with contact

TABLE 2. Containment diagram for the orbit closures of $SL(8)$ on $\mathbb{P}(\bigwedge^3 \mathbb{C}^8)$ together with their affine dimensions.



locus of projective dimension c is

$$(2) \quad \text{codim}(\sigma_k(X)^\vee) = k(c+1).$$

This leads us to the following two possibilities:

- if $\sigma_3(Gr(\mathbb{P}^2, \mathbb{P}^7))$ was the closure of XV then its dual variety would have codimension $3(c+1) = 56 - 40 = 16$, but this is impossible because c has to be a natural number;
- If $\sigma_3(Gr(\mathbb{P}^2, \mathbb{P}^7))$ is the closure of XIX then its dual variety has codimension $3(c+1) = 56 - 32 = 24$, this is clearly possible and it is the only possibility left.

This shows that $\sigma_3(Gr(\mathbb{P}^2, \mathbb{P}^7))^\vee = \sigma_2(Gr(\mathbb{P}^2, \mathbb{P}^7))$. Remark that this also shows that $Gr(\mathbb{P}^2, \mathbb{P}^7)$ is 2 and 3-weakly-defective and the dimension of the contact loci of $\sigma_2(Gr(\mathbb{P}^2, \mathbb{P}^7))$ and of $\sigma_3(Gr(\mathbb{P}^2, \mathbb{P}^7))$ are 3 and 7 respectively. \square

Question 1.4. It would be newsworthy to give a geometric description of the duality $\sigma_3(Gr(\mathbb{P}^2, \mathbb{P}^7))^\vee = \sigma_2(Gr(\mathbb{P}^2, \mathbb{P}^7))$. An interesting fact for this purpose is that $\sigma_2(Gr(\mathbb{P}^2, \mathbb{P}^7))$ is equal to the so called subspace variety $Sub_6(\wedge^3 \mathbb{C}^8) := \{t \in \mathbb{P}(\wedge^3 \mathbb{C}^8) \mid \exists \mathbb{C}^6 \subset \mathbb{C}^8 \text{ s.t. } t \in \mathbb{P}(\wedge^3 \mathbb{C}^6)\}$ (cfr [17, Ex. 7.1.4.3]). One containment is obvious and it holds for any secant variety of any Grassmannian with the correct adjusting of indices, the other containment is a peculiarity of this specific case.

Remark 1.5. As already remarked, the projective duality of in Table 1 is performed via the computation of the dimensions of the dual varieties of the orbit closure of any generator, and via a specific argument for XV and XIX given in the proof of Proposition 1.2. It is worth to remark that this duality almost corresponds to the duality of the arithmetic characters showed in [15]: they agree for almost all cases except for VI and X that are projectively dual to each other according to our computations, while Gurevich in [15, VII, §35.4] explicitly writes that those two orbits don't have any dual orbit. This is a very interesting and peculiar phenomenon that the projective duality does not correspond to the duality of arithmetic characters.

Corollary 1.6. *The variety $Gr(\mathbb{P}^2, \mathbb{P}^7)$ is 2 and 3-weakly defective.*

Proof. The duality $\sigma_3(Gr(\mathbb{P}^2, \mathbb{P}^7))^\vee = \sigma_2(Gr(\mathbb{P}^2, \mathbb{P}^7))$ together with the formula (2) (cfr. [8]) show that the contact locus of $\sigma_2(Gr(\mathbb{P}^2, \mathbb{P}^7))$ has dimension 3 and the contact locus of $\sigma_3(Gr(\mathbb{P}^2, \mathbb{P}^7))$ has dimension 7. \square

As already recalled in the introduction, the weakly defectiveness is not sufficient to claim anything about the identifiability.

Proposition 1.7. *The Grassmannian $Gr(\mathbb{P}^2, \mathbb{P}^7)$ is 3-identifiable.*

Proof. We computed with Macaulay2 ([14]) the tangentially contact locus at three points of $Gr(\mathbb{P}^2, \mathbb{P}^7)$; it turns out to be the union of three disjoint \mathbb{P}^3 's, each one passing through one and only one of the tangent points, and a \mathbb{P}^5 not passing to any one of the three points of tangency.

More precisely, the three points that we chose (before the Plücker embedding) were the following:

$$q_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad q_2 = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \end{pmatrix},$$

$$q_3 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

We computed the contact locus of the span of the three tangent spaces at these points. We found that in the coordinates $\{a_0, a_{1,1}, \dots, a_{3,5}\}$ of the \mathbb{P}^{15} parameterizing $Gr(\mathbb{P}^2, \mathbb{P}^7)$ the ideal of the contact locus is the intersection of the following four ideals:

$$\begin{aligned} I(\Pi_1) &= (a_{3,4}, a_{1,4}, a_{2,4}, a_{3,1}, a_{1,1}, a_{2,1}, a_{3,2}, a_{1,2}, a_{2,2}, a_{3,5}, a_{1,5}, a_{2,5}), \\ I(\Pi_2) &= (a_{3,4}, a_{1,4}, a_{2,4}-1, a_{3,1}, a_{1,1}, a_{2,1}, a_{3,2}, a_{1,2}, a_{2,2}, a_{3,3}+a_{3,5}-1, a_{1,3}+a_{1,5}-1, a_{2,3}+a_{2,5}), \\ I(\Pi_3) &= (a_{3,4}, a_{1,4}, a_{2,4}, a_{2,1}+a_{2,3}, a_{3,1}+a_{3,3}-1, a_{1,1}+a_{1,3}-1, a_{3,2}, a_{1,2}, a_{2,2}-1, a_{3,5}, a_{1,5}, a_{2,5}), \\ I(\Pi_4) &= (a_{3,4}, a_{1,4}, a_{3,1}, a_{1,1}, a_{3,3}-1, a_{1,3}-1, a_{3,2}, a_{1,2}, a_{3,5}, a_{1,5}). \end{aligned}$$

Clearly all the Π_i 's are linear and it is very easy to check that they remain linear even after the Plücker embedding $p_{3,8} : \mathbb{P}^{15} \rightarrow \mathbb{P}^{55}$. Moreover $\Pi_i \simeq p_{3,8}(\Pi_i) \simeq \mathbb{P}^3$ for $i = 1, 2, 3$ and $\Pi_4 \simeq p_{3,8}(\Pi_4) \simeq \mathbb{P}^5$. It's again an easy check that $q_i \in \Pi_i$ for $i = 1, 2, 3$ and that $q_i \notin \Pi_j$ for $i \neq j$, $i = 1, 2, 3$ and $j = 1, 2, 3, 4$. Remark also that the three \mathbb{P}^3 's have no common components. As already recalled in the Introduction, the fact that the 3-tangentially contact locus is linear in the Plücker space suffices to claim the 3-identifiability of $Gr(\mathbb{P}^2, \mathbb{P}^7)$ (cfr [8]). \square

Corollary 1.8. *The Grassmannian $Gr(\mathbb{P}^2, \mathbb{P}^7)$ is 2-identifiable.*

Proof. By definition of r -identifiability if X is r -identifiable then it is also $(r - k)$ -identifiable for any $0 \leq k \leq r - 1$. \square

Remark 1.9. We like to point out a very peculiar phenomenon that we have not found before in the literature. In the computation of the 3-tangentially contact locus of $Gr(\mathbb{P}^2, \mathbb{P}^7)$ (in the proof of Proposition 1.7) we found four components: three of them pass through the points of tangency, while the other one doesn't pass through any one of them.

Remark 1.10. The fact that $\sigma_2(Gr(\mathbb{P}^2, \mathbb{P}^7))$ and $\sigma_3(Gr(\mathbb{P}^2, \mathbb{P}^7))$ are but their generic element is identifiable is not a new phenomenon: in [10, Lemma 2.5 and Theorem 2.7] it is shown an analogous example in a case of secant varieties to Segre varieties.

1.2. Two decompositions for the generic element of $\sigma_5(Gr(\mathbb{P}^2, \mathbb{P}^9))$. We have computed with Macaulay2 [14] that $\sigma_5(Gr(\mathbb{P}^2, \mathbb{P}^9))$ has a positive dimensional contact locus with “very high probability”, i.e. that it should be weakly-defective. Here we want to prove that this is actually the case and moreover we can also show that its generic element is not identifiable. More precisely we prove the following result.

Proposition 1.11. *The generic element of $\sigma_5(Gr(\mathbb{P}^2, \mathbb{P}^9))$ has exactly 2 decompositions as sum of 5 points in $Gr(\mathbb{P}^2, \mathbb{P}^9)$.*

Proof. First of all we want to prove that the contact locus of $\sigma_5(Gr(\mathbb{P}^2, \mathbb{P}^9))$ is an elliptic curve obtained by the osculating 2-planes to an elliptic normal curve of degree 10 in \mathbb{P}^9 .

The Hilbert scheme of the elliptic normal curves of degree d in \mathbb{P}^{d-1} has dimension d^2 . If we consider 5 planes in \mathbb{P}^9 the double tangency of the curve with a \mathbb{P}^2 imposes 20 conditions ($\dim(Gr(\mathbb{P}^2, \mathbb{P}^9)) = 21$ and the elliptic curve must pass through one of its points) and in fact $20 \cdot 5 = 100 = d^2$, therefore we expect a finite number of elliptic curves such that five of their 2-osculating planes are assigned. Now, for a random case with Macaulay2 ([14]), we obtain only one elliptic curve,

therefore, by semicontinuity, we can say that the contact locus is given by only one elliptic curve.

In order to show that having an elliptic curve as a contact locus leads to exactly two decompositions for the generic element of $\sigma_5(Gr(\mathbb{P}^2, \mathbb{P}^9))$ it would be sufficient to quote [8]. Anyway, for the present specific example this can be shown geometrically. The contact locus curve \mathcal{C} that we have found with Macaulay2 ([14]) is an elliptic curve of degree 10 before being embedded with Plücker. By Hurwitz formula for curves, the degree of \mathcal{C} remains 10 even after the Plücker embedding. Now fix 5 points on \mathcal{C} and take all the (\mathbb{P}^8) 's containing them; they define a linear series and they intersect \mathcal{C} in other 5 points, moreover the two (\mathbb{P}^4) 's spanned by those two quintuple of points must intersect each other since they live in the same \mathbb{P}^8 . This is again sufficient to conclude that we have exactly two decompositions for the generic element of $\sigma_5(Gr(\mathbb{P}^2, \mathbb{P}^9))$. \square

Corollary 1.12. *The non r -defective Grassmannians $Gr(\mathbb{P}^k, \mathbb{P}^n)$ for $n < 14$ are all not r -weakly defective except for:*

- (a) $r = 2, 3$ and $Gr(\mathbb{P}^2, \mathbb{P}^7) \simeq Gr(\mathbb{P}^4, \mathbb{P}^7)$, where the contact loci have dimensions 3 and 7 respectively;
- (b) $r = 5$ and $Gr(\mathbb{P}^2, \mathbb{P}^9) \simeq Gr(\mathbb{P}^6, \mathbb{P}^9)$, where the contact locus has dimension 1;
- (c) $r = 2$ and $Gr(\mathbb{P}^2, \mathbb{P}^6) \simeq Gr(\mathbb{P}^3, \mathbb{P}^6)$, where the contact locus has dimension 6.

Proof. Case (a) is Corollary 1.6. The dimensions of the contact loci are computed in the proof of Proposition 1.2 when we show that $\sigma_2(Gr(\mathbb{P}^2, \mathbb{P}^7))$ and $\sigma_3(Gr(\mathbb{P}^2, \mathbb{P}^7))$ are dual to each other.

In the proof of Proposition 1.11 we showed that $Gr(\mathbb{P}^2, \mathbb{P}^9)$ is 5-tangentially weakly defective, therefore it is also 5-weakly defective. In the same proof we also showed that the contact locus is an elliptic normal curve. This proves case (b). As already said in the proof of Theorem 1.1, the fact that $Gr(\mathbb{P}^2, \mathbb{P}^9)$ is not 4-weakly defective is done by direct computation.

The only case that we have not proved jet is (c). We computed, with Macaulay2, the dimension of $(\sigma_2(Gr(\mathbb{P}^2, \mathbb{P}^6)))^\vee$, by considering $\sigma_2(Gr(\mathbb{P}^2, \mathbb{P}^6))$ to be the orbit of $e_0 \wedge e_1 \wedge e_2 + e_3 \wedge e_4 \wedge e_5$ via the action of $SL(7)$ in $\bigwedge^3 \mathbb{C}^7$. It turns out that $\dim(\sigma_2(Gr(\mathbb{P}^2, \mathbb{P}^6)))^\vee = 21$, therefore, by the displayed formula (2) above, the contact locus has dimension 6.

The fact that all the regular cases (i.e. Grassmannians with r -secant varieties of the expected dimension) not listed above are not weakly defective is a consequence of the computation that we have done in the proof of Theorem 1.1 that shows that in those cases all contact loci are 0-dimensional. \square

Corollary 1.13. *The non r -defective Grassmannians $Gr(\mathbb{P}^k, \mathbb{P}^n)$ for $n < 14$ are all not r -tangentially weakly defective except for*

- (1) $r = 3$ and $Gr(\mathbb{P}^2, \mathbb{P}^7)$, where the contact locus has dimension 5;
- (2) $r = 5$ and $Gr(\mathbb{P}^2, \mathbb{P}^9)$, where the contact locus has dimension 1.

Proof. Since the r -tangentially weakly defectiveness implies the r -weakly defectiveness, we have to check only weakly defective cases listed in Corollary 1.12.

We computed with Macaulay2 that the 2-tangentially contact locus of $Gr(\mathbb{P}^2, \mathbb{P}^7)$ is 0-dimensional. This suffice to prove that $Gr(\mathbb{P}^2, \mathbb{P}^7)$ is not 2-tangentially weakly defective.

In Proposition 1.7 we computed the 3-tangentially contact locus of $Gr(\mathbb{P}^2, \mathbb{P}^7)$ and we found that it is the union of three \mathbb{P}^3 's and a \mathbb{P}^5 .

In Proposition 1.11 we showed that the 5-th secant degree of $Gr(\mathbb{P}^2, \mathbb{P}^9)$ is two, therefore we don't have the identifiability of the generic element of $\sigma_2(Gr(\mathbb{P}^2, \mathbb{P}^9))$, hence $Gr(\mathbb{P}^2, \mathbb{P}^9)$ is 2-tangentially weakly defective. Moreover since the 5-contact locus has dimension 1, and the 5-tangentially contact locus has positive dimension, we can conclude that also the 5-tangentially contact locus has dimension 1.

In the proof of Theorem 1.1 we already computed with Macaulay2 that the 2-tangentially contact locus of $Gr(\mathbb{P}^2, \mathbb{P}^6)$ is 0-dimensional. This suffice to prove that $Gr(\mathbb{P}^2, \mathbb{P}^6)$ is not 2-tangentially weakly defective. \square

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